# On the Positive Pell Equation $y^2 = 20x^2 + 16$

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**Abstract – The binary quadratic equation represented by the pellian $y^2 = 20x^2 + 16$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.**

**Index Terms – Binary quadratic, hyperbola, parabola, integral solutions, Pell equation 2017 Mathematics subject classification: 11D09.**

## 1. INTRODUCTION

Diophantine equation of the form$y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equations that have interesting mathematicians all over the entire world, since antiquity. J.L Lagrange proved that all positive pell equation$y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions. In [1-4], an elementary proof of a ceriterium for the solvability of pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented.

In this communication, the positive pell equation is given by $y^2 = 20x^2 + 16$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solution are presented.

## 2. METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is

$$y^2 = 20x^2 + 16 \qquad (1)$$

Whose smallest positive integer solution is

$$x_0 = 1, y_0 = 6$$

To obtain the other solutions of (1), consider the pell equation

$$y^2 = 20x^2 + 1, \left(\widetilde{x_0}, \widetilde{y_0}\right) = (2,9), D = 20$$

Whose general solutions is given by

$$\widetilde{x_n} = \frac{1}{2\sqrt{20}} g_n, \ \widetilde{y_n} = \frac{1}{2} f_n$$

Where

$$f_n = (9 + 2\sqrt{20})^{n+1} + (9 - 2\sqrt{20})^{n+1}$$

$$g_n = (9 + 2\sqrt{20})^{n+1} - (9 - 2\sqrt{20})^{n+1}$$

Applying brahmagupta lemma between $(\widetilde{x_0}, \ \widetilde{y_0})$and, $(\widetilde{x_n}, \widetilde{y_n})$the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{3}{\sqrt{20}} g_n$$

$$y_{n+1} = 3f_n + \frac{10}{\sqrt{20}} g_n$$

The recurrence relations satisfied by $x$ and $y$ are given by

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the following table below:

| N | $x_{n+1}$ | $y_{n+1}$ |
|---|---|---|
| -1 | 1 | 6 |
| 0 | 21 | 94 |
| 1 | 377 | 1686 |
| 2 | 6765 | 30254 |
| 3 | 121393 | 542886 |

From the above table, we observe some interesting relations among the solutions which are presented below:

Each of the following expressions is a nasty number

- $\frac{6}{32}(12x_{2n+3} - 188x_{2n+2} + 64)$
- $\frac{6}{576}(12x_{2n+4} - 3372x_{2n+2} + 1152)$

- $\frac{6}{144}(12y_{2n+3}-840x_{2n+2}+288)$
- $\frac{6}{2576}(188y_{2n+4}-15080x_{2n+2}+5152)$
- $\frac{6}{32}(188x_{2n+4}-3372x_{2n+3}+64)$
- $\frac{6}{144}(188y_{2n+2}-40x_{2n+3}+288)$
- $\frac{6}{16}(188y_{2n+3}-840x_{2n+3}+32)$
- $\frac{6}{144}(188y_{2n+4}-15080x_{2n+3}+288)$
- $\frac{6}{2576}(3372y_{2n+2}-40x_{2n+4}+5152)$
- $\frac{6}{144}(3372y_{2n+3}-840x_{2n+4}+288)$
- $\frac{6}{16}(3372y_{2n+4}-15080x_{2n+4}+32)$
- $\frac{6}{32}(42y_{2n+2}-2y_{2n+3}+64)$
- $\frac{6}{576}(754y_{2n+1}-2y_{2n+4}+1152)$
- $\frac{6}{32}(754y_{2n+3}-42y_{2n+4}+64)$

Each of the following expressions is a perfect square

- $\frac{1}{32}(12x_{2n+3}-188x_{2n+2}+64)$
- $\frac{1}{576}(12x_{2n+4}-3372x_{2n+2}+1152)$
- $\frac{1}{144}(12y_{2n+3}-840x_{2n+2}+288)$
- $\frac{1}{2576}(9y_{2n+4}-15080x_{2n+2}+5152)$
- $\frac{1}{32}(188x_{2n+4}-3372x_{2n+3}+64)$
- $\frac{1}{144}(188y_{2n+2}-40x_{2n+3}+288)$
- $\frac{1}{16}(188y_{2n+3}-840x_{2n+3}+32)$
- $\frac{1}{144}(188y_{2n+4}-15080x_{2n+3}+288)$
- $\frac{1}{2576}(3372y_{2n+2}-40x_{2n+3}+5152)$
- $\frac{1}{144}(3372y_{2n+3}-840x_{2n+4}+288)$
- $\frac{1}{16}(3372y_{2n+4}-15080x_{2n+4}+32)$
- $\frac{1}{32}(42y_{2n+2}-2y_{2n+3}+64)$
- $\frac{1}{576}(754y_{2n+1}-2y_{2n+4}+1152)$
- $\frac{1}{32}(754y_{2n+3}-42y_{2n+4}+64)$

Each of the following expressions is a cubical integer

- $\frac{1}{32}(12x_{3n+4}-188x_{3n+3})+3f_n$
- $\frac{1}{576}(12x_{3n+4}-3372x_{3n+3})+3f_n$
- $\frac{1}{144}(12y_{3n+4}-840x_{3n+3})+3f_n$
- $\frac{1}{2576}(12y_{3n+5}-15080x_{3n+3})+3f_n$
- $\frac{1}{32}(188x_{3n+5}-3372x_{3n+4})+3f_n$
- $\frac{1}{144}(188y_{3n+3}-40x_{3n+4})+3f_n$
- $\frac{1}{16}(188y_{3n+4}-840x_{3n+4})+3f_n$
- $\frac{1}{144}(188y_{3n+5}-15080x_{3n+4})+3f_n$
- $\frac{1}{2576}(3372y_{3n+3}-40x_{3n+5})+3f_n$
- $\frac{1}{144}(3372y_{3n+4}-840x_{3n+5})+3f_n$
- $\frac{1}{16}(3372y_{3n+5}-15080x_{3n+5})+3f_n$
- $\frac{1}{32}(42y_{3n+3}-2y_{3n+4})+3f_n$
- $\frac{1}{576}(754y_{3n+3}-2y_{3n+5}+3f_n$
- $\frac{1}{32}(754y_{3n+4}-42y_{3n+5})+3f_n$

From relation satisfied by the solutions are follows

- $64x_{n+3}=1345364x_{n+1}-62916x_{n+2}$
- $64y_{n+1}=32x_{n+2}-288x_{n+1}$
- $64y_{n+2}=288x_{n+2}-32x_{n+1}$
- $64y_{n+3}=5152x_{n+2}-288x_{n+1}$
- $1152x_{n+2}=64x_{n+3}-64x_{n+1}$
- $1152y_{n+1}=32x_{n+3}-5152x_{n+1}$
- $1152y_{n+2}=288x_{n+3}-288x_{n+1}$
- $1152y_{n+3}=5152x_{n+3}-32x_{n+1}$
- $288y_{n+1}=32y_{n+2}-1280x_{n+1}$
- $288y_{n+3}=5152y_{n+2}+1280x_{n+1}$
- $5152y_{n+1}=32y_{n+3}-23040x_{n+1}$
- $64y_{n+1}=288x_{n+3}-5152x_{n+2}$
- $64y_{n+2}=32x_{n+3}-288x_{n+2}$
- $64y_{n+3}=288x_{n+3}-32x_{n+2}$
- $288x_{n+1}=32x_{n+2}-64y_{n+1}$
- $288y_{n+2}=32y_{n+1}+1280x_{n+2}$

- $288y_{n+3} = 288y_{n+1} + 23040x_{n+2}$
- $32y_{n+3} = 288y_{n+2} + 1280x_{n+2}$
- $5152y_{n+2} = 288y_{n+1} + 1280x_{n+3}$
- $5152y_{n+3} = 32y_{n+1} + 23040x_{n+3}$
- $288x_{n+2} = 32x_{n+3} - 64y_{n+2}$
- $288y_{n+3} = 32y_{n+2} + 1280x_{n+3}$
- $32x_{n+2} = 288x_{n+3} - 64y_{n+3}$
- $64x_{n+2} = 1086y_{n+2} - 16790y_{n+1}$
- $64x_{n+3} = 19478y_{n+2} - 301134y_{n+1}$
- $64y_{n+3} = 1152y_{n+2} - 64y_{n+1}$
- $1152x_{n+1} = 70y_{n+3} - 19478y_{n+1}$
- $1152x_{n+2} = 1086y_{n+3} - 301134y_{n+1}$
- $1152x_{n+3} = 19478y_{n+3} - 5400934y_{n+1}$
- $1152y_{n+2} = 64y_{n+1} - 64y_{n+3}$
- $64x_{n+1} = 1086y_{n+3} - 19478y_{n+2}$
- $64x_{n+2} = 16790y_{n+3} - 301134y_{n+2}$
- $64x_{n+3} = 301134y_{n+3} - 5400934y_{n+2}$

## 3. REMARKABLE OBSERVATIONS

I.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below:

| S.No | Hyperbola | $X, Y$ |
|---|---|---|
| 1. | $20X^2 - Y^2 = 81920$ | $(12x_{n+2} - 188x_{n+1})$ $(840x_{n+1} - 40x_{n+2})$ |
| 2. | $X^2 - 20Y^2 = 1327104$ | $(12x_{n+3} - 3372x_{n+1})$ $(754x_{n+1} - 2x_{n+3})$ |
| 3 | $X^2 - 20Y^2 = 82944$ | $(12y_{n+2} - 840x_{n+1})$ $(188x_{n+1} - 2y_{n+2})$ |
| 4. | $X^2 - 20Y^2 = 26543104$ | $(12y_{n+3} - 15080x_{n+1})$ $(3372x_{n+1} - 2y_{n+3})$ |
| 5. | $X^2 - 20Y^2 = 4096$ | $(188x_{n+3} - 3372x_{n+2})$ $(754x_{n+2} - 42x_{n+3})$ |
| 6. | $X^2 - 20Y^2 = 82944$ | $(188y_{n+1} - 40x_{n+2})$ $(12x_{n+2} - 42y_{n+1})$ |
| 7. | $X^2 - 20Y^2 = 1024$ | $(188y_{n+2} - 840x_{n+2})$ $(12x_{n+2} - 42y_{n+2})$ |
| 8. | $X^2 - 20Y^2 = 82944$ | $(188y_{n+3} - 15080x_{n+2})$ $(3372x_{n+2} - 42y_{n+3})$ |
| 9. | $X^2 - 20Y^2 = 26543104$ | $(3372y_{n+1} - 40x_{n+3})$ $(12x_{n+3} - 754y_{n+1})$ |
| 10. | $X^2 - 20Y^2 = 82944$ | $(3372y_{n+2} - 840x_{n+3})$ $(188x_{n+3} - 754y_{n+2})$ |
| 11. | $X^2 - 20Y^2 = 1024$ | $(3372y_{n+3} - 15080x_{n+3})$ $(3372x_{n+3} - 754y_{n+3})$ |
| 12. | $20X^2 - Y^2 = 81920$ | $(42y_{n+1} - 2y_{n+2})$ $(12y_{n+2} - 188y_{n+1})$ |
| 13. | $20X^2 - Y^2 = 26542080$ | $(754y_{n+1} - 2y_{n+3})$ $(12y_{n+3} - 3372y_{n+1})$ |
| 14. | $20X^2 - Y^2 = 81920$ | $(754y_{n+2} - 2y_{n+3})$ $(188y_{n+3} - 3372y_{n+2})$ |

II.Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table below:

| S.No | Parabola | X,Y |
|---|---|---|
| 1. | $Y^2 = 640X - 81920$ | $(12x_{2n+3} - 188x_{2n+2} + 64)$ $(840x_{n+1} - 40x_{n+2})$ |
| 2. | $Y^2 = 576X - 1327104$ | $(12x_{2n+4} - 3372x_{2n+2} + 1152)$ $(754x_{n+1} - 2x_{n+3})$ |
| 3. | $Y^2 = 144X - 82944$ | $(12y_{2n+3} - 840x_{2n+2} + 288)$ $(188x_{n+1} - 2y_{n+2})$ |
| 4. | $20Y^2 = 2576X - 26543104$ | $(12y_{2n+4} - 15080x_{2n+2} + 5152)$ $(3372x_{n+1} - 2y_{n+3})$ |
| 5. | $20Y^2 = 32X - 4096$ | $(188x_{2n+4} - 3372x_{2n+3} + 64)$ $(754x_{n+2} - 42x_{n+3})$ |
| 6. | $20Y^2 = 144X - 82944$ | $(188y_{2n+2} - 40x_{2n+3} + 288)$ $(12x_{n+2} - 42y_{n+1})$ |
| 7. | $20Y^2 = 16X - 1624$ | $(188y_{2n+3} - 840x_{2n+3} + 32)$ $(188x_{n+2} - 42y_{n+2})$ |
| 8. | $20Y^2 = 144X - 82944$ | $(188y_{2n+4} - 40x_{2n+3} + 288)$ $(3372x_{n+2} - 42y_{n+3})$ |
| 9. | $20Y^2 = 2176X - 26543104$ | $(3372y_{2n+2} - 40x_{2n+4} + 5152)$ $(188x_{n+3} - 754y_{n+1})$ |
| 10. | $20Y^2 = 144X - 32944$ | $(3372y_{2n+3} - 340x_{2n+4} + 288)$ $(188x_{n+3} - 754y_{n+2})$ |
| 11. | $20Y^2 = 16X - 1024$ | $(3372y_{2n+4} - 15080x_{2n+4} + 32)$ $(3372x_{n+3} - 754y_{n+3})$ |
| 12. | $Y^2 = 640X - 81920$ | $(42y_{2n+2} - 2y_{2n+3} + 64)$ $(12y_{n+2} - 188y_{n+1})$ |
| 13. | $Y^2 = 576X - 26542080$ | $(754y_{2n+2} - 2y_{2n+4} + 1152)$ $(12y_{n+3} - 3372y_{n+1})$ |
| 14. | $Y^2 = 640X - 81920$ | $(754y_{2n+3} - 42y_{2n+4} + 64)$ $(527y_{n+3} - 57965y_{n+2})$ |

## 4. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation

$y^2 = 20x^2 + 16$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive pell equations and determine their integer solutions along with suitable properties.

## REFERENCES

[1] Dickson LE; History of theory of Number. Volume 2, Chelsea publishing company, New York, 1952.
[2] Mordel. L. J, Diophantine equations. Academic press, New York, 1969.
[3] Telang s, Number theory. Tata Mcgraw Hill publishing company Ltd., New Delhi, 2000.
[4] Burton D; Elementary Number theory. Tata McGraw Hill publishing company, New Delhi, 2002
[5] Goplan MA, Janaki G; observation on $y^2 = 3x^2 + 1$ . Acta cinacia indicia, 2008; XXXIVM (2):693-696.
[6] Goplan MA, Sangeetha G;A Remarkable observation on $y^2 = 10x^2 + 1$.Impact J Sci Tech., 2010;103-106.
[7] Goplan MA, palanikumar R; observation on$y^2 = 12x^2 + 1$. Antarctica J Math., 2011; 8(2):149-152.
[8] Gopalan MA, Vidhyalakshimi S, devibalaSon the Diophantine equation $3x^2 + xy = 14$acta cianciaindica, 2007; XXIIIM (2):645-648.
[9] Gopalan MA, Vijayalakshimi R; observation on the integral solution of$y^2 = 5x^2 + 1$. Impact J Sci Tech., 2010; 4:125-129.
[10] Gopalan MA, Yamuna RS; Remarkable observation on the binary quadratic equation $y^2 = k^2 + 2x^2 + 1, k£ Z - \{0\}$. Impact J Sci Tech., 2010;4:61-65
[11] Gopalan MA, Sivagami B; observation on the integral solutions of$y^2 = 7x^2 + 1$. Antarctica J Math., 2010; 7(5):503-507.
[12] Gopalan MA, Vidhyalakshimi R; Special Pythagorean triangle generated through the integral solutions of the equation$y^2 = k^2 + 1x^2 + 1$. Antarctica J Math 7(5):503-507.
[13] Gopalan MA, Srividhya G; Relation among M-goal Number through theequation$y^2 = 2x^2 + 1$ . Antarctica J Math 7(3):363-369.
[14] Gopalan MA, Vidhyalakshimi S,Usharani TR,Malika S;observation on $y^2 = 2x^2 - 3$.Bessel.,J Math.,2010;2(3):153-158.
[15] Gopalan MA, Vidhyalakshimi S,Umarani J; Remarkable observations on the hyperbola $y^2 = 24x^2 + 1$.Bulletin of Mathematics and Statistics Research, 2013;1:9-12.
[16] Gopalan MA, Vidhyalakshimi S,Maheswari D; Remarkable observations on the hyperbola $y^2 = 30x^2 + 1$. International Journal of Engineering of Research, 2013; 1(3):312-314.